

# Cosmological Topologically Massive Gravitons and Photons<sup>\*</sup>

S. CARLIP<sup>\*</sup>, S. DESER<sup>‡</sup>, A. WALDRON<sup>†</sup>, AND D. K. WISE<sup>†</sup>

<sup>\*</sup>*Department of Physics, University of California, Davis, CA 95616, USA*  
carlip@physics.ucdavis.edu

<sup>‡</sup>*Lauritsen Laboratory, California Institute of Technology, Pasadena, CA 91125*  
and  
*Department of Physics, Brandeis University, Waltham, MA 02454, USA*  
deser@brandeis.edu

<sup>†</sup>*Department of Mathematics, University of California, Davis, CA 95616, USA*  
wally,derek@math.ucdavis.edu

JUNE 18, 2008

## ABSTRACT

We study topologically massive (2+1)-dimensional gravity with a negative cosmological constant. The masses of the linearized curvature excitations about  $\text{AdS}_3$  backgrounds are not only shifted from their flat background values but, more surprisingly, split according to chirality. For all finite values of the topological mass, we find a single bulk degree of freedom with positive energy, and exhibit a complete set of normalizable, finite-energy wave packet solutions. This model can also be written as a sum of two higher-derivative  $\text{SL}(2, \mathbb{R})$  Chern–Simons theories, weighted by the central charges of the boundary conformal field theory. At two particular ‘critical’ values of the couplings, one of these central charges vanishes, and linearized topologically massive gravity becomes equivalent to topologically massive electromagnetism; however, the physics of the bulk wave packets remains unaltered here.

---

<sup>\*</sup>Some of these results were reported in abbreviated form in [1].

# 1 Introduction

Topologically massive theories [2] are three-dimensional gauge invariant systems with actions consisting of the normal kinetic term plus a Chern-Simons term, and come in both vector (abelian or Yang-Mills) and tensor (gravitational) versions. These theories' excitations were found to have rather surprising complementary properties: they are both massive and gauge invariant, and their single, parity-violating, degrees of freedom are represented by indexless “scalar” fields with nonvanishing spins. Further, the energy of the linearized gravitons is manifestly positive if and only if the sign of the Einstein term is taken to be the opposite of the usual one in  $d = 4$ .

While these results are quite old, the obvious cosmological completion of topologically massive gravity—the addition of a cosmological constant—was only studied more recently [3–6]. However, its supersymmetric (supergravity) extension was given long ago [7], and has the powerful corollary that the underlying bosonic model (with the above sign!) has positive energy—which is still defined in an anti-de Sitter context [8]—because of the usual  $E = \{Q, Q^\dagger\}$  SUSY relation. (As in all supergravities, a negative cosmological constant is required.) Similar considerations hold for the graviton's spin 3/2 partner [9]. Cosmological topologically massive gravity is the system we will study here, with a warm-up via scalars and cosmological topologically massive electrodynamics.

As is well known, fields in (anti-) de Sitter backgrounds can behave quite differently from those in flat spacetime. Indeed, some of our results will be reminiscent of those obtained earlier [10, 11] for “partially massive” tensor models in  $d = 4$ . The effective masses of our excitations are shifted, but a suitable tuning in the  $(m^2, \Lambda)$  plane simultaneously keeps a single degree of freedom while allowing propagation on the null cone. Importantly, what remains unchanged is the single degree of freedom and its description via an indexless “scalar” field. This field has a mass that always respects the Breitenlohner–Freedman bound, which allows an extended range of negative squared-masses  $m^2 \geq \Lambda$  for fields in anti-de Sitter space [12, 13]. It is thus a consistent field theory. We demonstrate this result both in light-front gauge and in terms of manifestly gauge-invariant variables.

Perhaps even more interesting than the possibility of lightlike propagation for special tunings of the mass parameter is the chirality dependence of the mode masses. The decay rate of metric fluctuations near the anti-de Sitter boundary also depends on their chirality. This has consequences for the dis-

cussion of boundary behavior in [6] and the conjecture that the bulk modes disappear and the boundary theory becomes chiral at a critical value  $\mu^2 = -\Lambda$  of the ‘topological mass’  $\mu$ . We find that at this critical point, the metric fluctuations can form finite-energy wave packets, with finite Fefferman–Graham asymptotics [14] and no constraints on (three-dimensional) chirality.\* For all smaller values of the mass parameter, however, the Fefferman–Graham expansion contains divergent terms, perhaps necessitating a truncation to a single handedness or addition of boundary counterterms to the action. Our results show that the mechanism behind this truncation is not a new gauge invariance. We stress that for generic masses—including the critical value—the theory describes a single, positive energy bulk mode with correct asymptotics. These results connect smoothly to existing Minkowski ones [2] as the cosmological constant is taken to zero. There also exist exact pp-wave chiral solutions at the critical point and at higher values of  $|\mu|$ .

As noted above, positive energy for the massive fluctuations requires the Einstein part of the action to have a sign opposite to the canonical choice for pure ( $d = 4$ ) gravity. This leads to no inconsistencies in flat vacua, since three-dimensional pure Einstein massless bulk gravitational modes are pure gauge. For a negative cosmological constant, however, this choice of sign also leads to negative energy for the BTZ black hole, which is an exact solution not only of pure Einstein gravity, but also of our field equations for arbitrary  $\mu$  [16]. This suggests that the theory may be fundamentally unstable. While this is a serious concern, instability is by no means certain: Classically, it is not clear that a negative-energy black hole can be created from positive-energy matter, and while the potential quantum instability may pose a more serious problem, we know of no instanton that mediates the production of negative-energy black holes. Indeed, the possibility remains that one can find a superselection sector in which BTZ black holes are excluded, much as one excludes negative-energy Schwarzschild black holes in 3+1 dimensions. The supergravity positive energy arguments of [7] provide corroborative evidence.

The alternative considered in [6] is to flip the overall sign of the action, keeping positive energy BTZ black holes, but at the price of negative-energy

---

\*There is an unfortunate semantic confusion coming from two different meanings of the term “chiral.” We are using the term here in the three-dimensional sense: our bulk modes depend on both  $x^+$  and  $x^-$  in light-front coordinates. Similarly, the solutions we discuss in section 9 are chiral in the sense that they depend, in the bulk, only on  $x^+$  and not  $x^-$ . This is quite distinct from the boundary chirality discussed in [6, 15], which is determined by global properties of the diffeomorphism generators.

bulk excitations. There it was argued that at the critical value  $\mu^2 = -\Lambda$ , the propagating bulk degrees of freedom become pure gauge, restoring the consistency of the theory. We fail to find this behavior; rather, we exhibit well-behaved propagating bulk modes with good asymptotic properties (but negative energy with the flipped sign of the action).

The critical value of  $\mu$  does have its own peculiarities, however. Like three-dimensional Einstein gravity [17, 18], the theory can be written for any  $\mu$  as a sum of two (higher-derivative) Chern–Simons theories. At  $\mu^2 = -\Lambda$ , this action degenerates to a single such Chern–Simons term. Separately, we demonstrate that linearized topologically massive gravity at this critical point is equivalent to topologically massive electrodynamics in an anti-de Sitter background. This again confirms that linearized topologically massive gravity describes a single, consistent field theoretic bulk degree of freedom, also at this point.

We will begin by reviewing scalar fields in  $\text{AdS}_3$ , focusing on the light-front formalism. We next discuss topologically massive photons in Section 3, reducing the dynamics to a single “scalar” mode in a light-front approach. We then rederive the same results in terms of manifestly gauge invariant curvatures, and display the splitting of masses according to the chirality of the excitation. In Section 4, we apply the same logic to linearized metric fluctuations, and show that cosmological massive gravitons can be described as “scalars,” again with masses that depend on their chirality. All of our masses satisfy the three-dimensional Breitenlohner–Freedman bounds [12, 13], and therefore describe positive energy excitations. In the next two sections, we study the asymptotics of these metric fluctuations and the construction of a complete set of normalizable, finite-energy wave packets. In Sections 7 and 8, we explore the Chern–Simons action at the critical topological mass, and reformulate the linearized graviton theory as topologically massive electrodynamics. In Section 9, we examine the chiral solutions and show that they are exact pp-waves. Finally, many detailed but useful calculations are relegated to an Appendix. These include explicit solutions for all of the bulk excitations, the scalar bulk-boundary intertwiner, and a simple rederivation of the Breitenlohner–Freedman bound.

## 2 AdS<sub>3</sub> Scalars

We denote by  $\gamma_{\mu\nu}$  the dynamical metric, with signature  $(-++)$ , reserving  $g_{\mu\nu}$  for the background AdS<sub>3</sub> metric and  $D$  for the background connection. Our other conventions are as follows: the Ricci tensor—equivalent in three dimensions to the full curvature—is

$$R_{\nu\tau} \equiv R^\sigma{}_{\nu\sigma\tau} = \partial_\sigma \Gamma^\sigma_{\tau\nu} - \partial_\tau \Gamma^\sigma_{\sigma\nu} + \Gamma^\sigma_{\sigma\mu} \Gamma^\mu_{\tau\nu} - \Gamma^\sigma_{\tau\mu} \Gamma^\mu_{\sigma\nu},$$

and the scalar curvature is the positive trace,  $R = R^\mu{}_\mu$ . In particular, for the AdS<sub>3</sub> background ( $\Lambda < 0$ ), we have

$$R_{\alpha\beta\mu\nu} = \Lambda(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \quad R_{\mu\nu} = 2\Lambda g_{\mu\nu} \quad \text{and} \quad R = 6\Lambda. \quad (2.1)$$

We employ units

$$\Lambda = -1; \quad (2.2)$$

the cosmological constant can always be reinstated by dimensional analysis.

In our primarily light-front approach, we use the Poincaré frame

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{2dx^+ dx^- + dz^2}{z^2}, \quad (2.3)$$

where  $x^\pm = \frac{1}{\sqrt{2}}(x \pm t)$ . We view  $x^+ \equiv \tau$  as a time coordinate and often denote  $\frac{\partial}{\partial\tau}$  by a dot or  $\partial_+$ . We also often write  $\partial$  for  $\frac{\partial}{\partial z}$  and  $\partial_-$  for  $\frac{\partial}{\partial x^-}$ .

While these choices greatly simplify many computations, the coordinates (2.3) do not cover the whole of anti-de Sitter space. Recall that AdS<sub>3</sub> is the simply connected covering space of the hyperboloid  $-U^2 - V^2 + X^2 + Y^2 = -1$ . A Poincaré patch covers half of this hyperboloid, say the region  $U + X > 0$ , and the coordinates (2.3) correspond to setting  $U + X = 1/z$ ,  $Y = x/z$ ,  $V = -t/z$ ,  $U - X = (-t^2 + x^2 + z^2)/z$ . Results in other Poincaré patches are easily obtained by successive inversion operations: for example, the patch  $X - U > 0$  is reached by the diffeomorphism  $x^\mu = -\tilde{x}^\mu / (\tilde{x}^\nu \eta_{\nu\rho} \tilde{x}^\rho)$ , under which the form of the metric is invariant.

The key idea of the light-front method is that actions are automatically first order, albeit with a nonstandard symplectic form given by the anti-symmetric bracket

$$\langle A, B \rangle \equiv \int dx^- dz A \partial_- B = -\langle B, A \rangle. \quad (2.4)$$

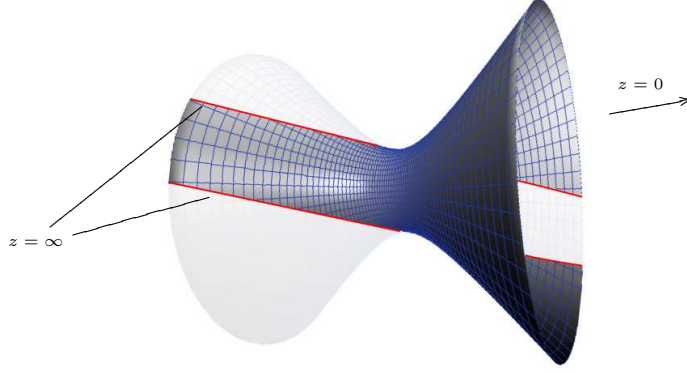


Figure 1: A 2d cross-section—corresponding to  $V = 0$  in the ambient four-dimensional coordinates, or equivalently  $t = 0$  in Poincaré coordinates—of the Poincaré patch, showing how it sits in the hyperboloid model of  $\text{AdS}_3$ . The surface  $z = 0$  lies in the  $\text{AdS}_3$  boundary. The  $z = \infty$  surface intersects any constant-Poincaré-time slice in a pair of lines through the bulk, as shown; these meet asymptotically (to the left, in the picture) in a single point on the boundary.

In particular, the standard (positive energy) massive scalar field action

$$I = -\frac{1}{2} \int d^3x \sqrt{-g} \left\{ \partial_\mu \varphi g^{\mu\nu} \partial_\nu \varphi + m^2 \varphi^2 \right\}, \quad (2.5)$$

rewritten in light-front coordinates, takes the Hamiltonian form

$$I = \int d\tau \left\{ \langle \phi, \dot{\phi} \rangle - \left( \frac{1}{2} [\partial \phi]^2 + \frac{1}{2z^2} \left[ m^2 + \frac{3}{4} \right] \phi^2 \right) dx^- dz \right\}, \quad (2.6)$$

where we have made the field redefinition

$$\phi \equiv \frac{1}{\sqrt{z}} \varphi. \quad (2.7)$$

Varying this action yields the equation of motion

$$\left[ 2\partial_- \partial_+ + \partial^2 - \frac{m^2 + 3/4}{z^2} \right] \phi = 0. \quad (2.8)$$

Setting  $m^2 = -3/4$  yields the massless Minkowski wave equation, which is hardly surprising, as the background is conformally flat, and moreover this value corresponds precisely to adding the conformal improvement term  $\frac{1}{16} R \varphi^2$

to the scalar action, which gives lightlike propagation [10]. The scaling (2.7) is of course the usual conformal one for a scalar in the metric (2.3). Note also that this negative value of  $m^2$  is consistent, since the Breitenlohner–Freedman bound in three dimensions allows for masses as low as  $m^2 = -1$  [12].

For generic values of the mass, this equation is most easily solved by going to Fourier space in the  $x^\pm$  coordinates. The expression  $2\partial_-\partial_+ = -\partial_t^2 + \partial_x^2$  then transforms to the frequency  $\omega^2 = E^2 - p_x^2$ , yielding Bessel’s equation in the variable  $z$  for the combination  $\varphi(z)/z$ ,

$$\left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} + \omega^2 - \frac{\nu^2}{z^2} \right] \left( \frac{\varphi}{z} \right) = 0, \quad (2.9)$$

with index

$$\nu^2 = m^2 + 1. \quad (2.10)$$

The solutions of (2.9) are oscillatory; indeed, when  $\nu$  is a half-integer, the Bessel function solution reduces to a slowly varying function of  $z$  multiplied by a plane wave. That is, when

$$m^2 + 1 = (1/2)^2, (3/2)^2, (5/2)^2, \dots \quad (2.11)$$

it follows that

$$\varphi \sim \left( \text{slowly varying} \right) \cdot \exp(ik_\mu x^\mu), \quad (2.12)$$

with  $k_\mu \eta^{\mu\nu} k_\nu = 0$ . Hence these values of the mass parameter imply lightlike propagation. Exactly this mechanism was found for all integer spin fields in  $d = 4$  cosmological backgrounds [11].

The Poincaré coordinate patch contains two pieces of the anti-de Sitter boundary, one at  $z = 0$  and a second line at  $z = \infty$  (see Figure 1). Demanding that solutions to the anti-de Sitter Klein–Gordon equation remain finite at  $z = 0$  requires solutions to be Bessel functions of the first kind,  $J_\nu$ , which behave as  $z^\nu$  at the boundary. At large  $z$ , the original scalar field  $\varphi$  then goes as  $z^{1/2}$  (up to an oscillatory factor), a potentially dangerous behavior at the part of the boundary with  $z = \infty$ .<sup>\*</sup> This difficulty can be handled by studying wave packets with support away from  $z \rightarrow \infty$ . We discuss this in more detail in Section 6.

Note that to obtain the Bessel function solutions given above, we implicitly assumed that  $\omega \neq 0$ , that is, that  $\partial_+\partial_-\varphi \neq 0$ . Additional chiral solutions,

---

<sup>\*</sup>We thank A. Strominger for this important observation.

for which  $\partial_+\varphi$  or  $\partial_-\varphi$  vanish, also occur. These pp-wave excitations will be discussed in Section 9. Further modified Bessel function solutions can be found by allowing  $\omega$  to become imaginary, that is, letting  $k > E$  [19]. The modified Bessel function of the second kind decays asymptotically for large  $z$ , but blows up as  $z^{-\nu}$  as  $z \rightarrow 0$ . One might hope that for small masses—and thus small values of the index  $\nu$ —such solutions might be admissible. Indeed, at the critical value  $\mu = 1$  these solutions behave logarithmically, much like the finite-energy solutions found in [20]. Unfortunately, however, in contrast to the asymptotically anti-de Sitter behavior of those solutions, the modified Bessel functions lead to metric components that diverge as  $z^{-2} \log z$ .

Our next step is to rewrite three-dimensional topologically massive electrodynamics and linearized gravity as scalar fields to which the above analysis applies. More precisely, these “scalars” are indexless (nonlocal) gauge invariant components of the corresponding spin-1 and -2 fields.

### 3 Topologically Massive AdS<sub>3</sub> Photons

We now turn to topologically massive electrodynamics in an anti-de Sitter background, with action

$$I = -\frac{1}{4} \int d^3x \left\{ \sqrt{-g} F_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} + \mu \varepsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho \right\}. \quad (3.1)$$

The parameter  $\mu$  has dimensions of mass. We write  $A_\mu = (A_+, A_-, A)$ , where  $A \equiv A_z$ , and employ the light-front coordinates and conventions of the preceding section. Since we are dealing with local bulk degrees of freedom, we may assume the operator  $\partial_-$  to be invertible. In that case, the field equation for  $A_+$ ,

$$\partial_- A_+ = \dot{A}_- - \frac{1}{z} [z\partial + \mu + 1] \varphi \quad (3.2)$$

with

$$\varphi \equiv A - \frac{\partial}{\partial_-} A_- , \quad (3.3)$$

is algebraically solvable and may be substituted back into the action, which becomes

$$I = \int d\tau \left\{ z \langle \varphi, \dot{\varphi} \rangle - \frac{1}{2z} ([z\partial + \mu + 1] \varphi)^2 dx^- dz \right\}. \quad (3.4)$$



To obtain the light-front symplectic form, we make the field redefinition

$$\phi = \sqrt{z} \varphi, \quad (3.5)$$

so that

$$I = \int d\tau \left\{ \langle \phi, \dot{\phi} \rangle - \left( \frac{1}{2} [\partial \phi]^2 + \frac{1}{2z^2} \left[ (\mu + 1)^2 - \frac{1}{4} \right] \phi^2 \right) dx^- dz \right\}. \quad (3.6)$$

We have thus achieved our goal: this is again the action (2.6) for a scalar field, with mass squared

$$m^2 = (\mu + 1)^2 - 1. \quad (3.7)$$

Observe that the Bessel index for solutions to the wave equation is

$$\nu^2 = (\mu + 1)^2, \quad (3.8)$$

so half-integer values of  $\mu$  lead to lightlike propagation. The Breitenlohner–Freedman bound is saturated at  $\mu = -1$ .

One observation will simplify our graviton analysis in the next section. The combination  $\varphi = A - (\partial/\partial_-)A_-$  is gauge invariant under  $\delta A = \partial\alpha$  and  $\delta A_- = \partial_- \alpha$ . Therefore we could have expedited our analysis by choosing the “light-front gauge”  $A_- = 0$ .

The mass formula (3.7) is apparently asymmetric under  $\mu \mapsto -\mu$ . The explanation for this is an interesting interplay between the chiral nature of cosmological topologically massive electrodynamics and the anti-de Sitter background. First, we identify

$$\partial_- \varphi = \partial_- A - \partial A_- = F_{-z} \equiv F_- \quad (3.9)$$

with the left component of the electromagnetic field strength. In flat space  $F_+$ ,  $F_-$  and  $F_{+-}$  are all propagating modes with equal masses. But our computations show that a negative cosmological constant splits the masses of these modes:

Field	$m^2$
$F_+$	$(\mu - 1)^2 - 1$
$F_{+-}$	$\mu^2 - 1$
$F_-$	$(\mu + 1)^2 - 1$

(3.10)

Note all values of  $\mu$  respect the Breitenlohner–Freedman bound  $m^2 \geq -1$ . These results are symmetric under a simultaneous flip of the sign of  $\mu$  and chirality.

As a check of our computations, and to exhibit that our conclusions in no way rely on the invertibility of  $\partial_-$  or any other light-front peculiarity, we rederive these results directly from the cosmological topologically massive electrodynamics equations of motion for the field strengths. Writing out the Bianchi identity

$$\partial_{[\mu} F_{\nu\rho]} = 0 \quad (3.11)$$

in the Poincaré coordinates (2.3) yields

$$\partial_+ F_- - \partial_- F_+ + \partial F_{+-} = 0, \quad D^\nu J_\nu = 0 \quad (3.12)$$

while the equations of motion

$$J_\nu = D^\mu F_{\mu\nu} - \frac{\mu}{2\sqrt{-g}} \varepsilon_\nu^{\mu\rho} F_{\mu\rho} = 0 \quad (3.13)$$

give

$$\partial_+ F_- + \partial_- F_+ = \frac{\mu}{z} F_{+-}, \quad (3.14)$$

$$\partial F_+ + \partial_+ F_{+-} = \frac{\mu - 1}{z} F_+, \quad (3.15)$$

$$\partial F_- - \partial_- F_{+-} = -\frac{\mu + 1}{z} F_- . \quad (3.16)$$

These equations are easily manipulated to give scalar wave equations\* for any of  $F_\pm$ ,  $F_{+-}$ : for example, taking  $\partial_-$  of the sum of (3.12) and (3.14) and using (3.16) to eliminate  $\partial_- F_{+-}$  yields

$$\left[ 2\partial_- \partial_+ + \partial^2 - \frac{(\mu + 1)^2 - 1/4}{z^2} \right] (\sqrt{z} F_-) = 0. \quad (3.17)$$

This is precisely the scalar wave equation of motion that follows from the scalar action (3.6) obtained through a light-front analysis. The same results hold for  $F_+$  and  $F_{+-}$ , but with masses as quoted in (3.10).

---

\*Of course, all components of  $F_{\mu\nu}$  can be derived from the gauge invariant quantity  $\varphi$  in (3.3) (see also the Appendix).

To conclude this section, we note that the energy in cosmological topologically massive electrodynamics is manifestly non-negative. Since the topological Chern-Simons term is metric-independent, the stress-energy tensor is simply

$$T_{\mu\nu} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \quad D^{\mu}T_{\mu\nu} = 0, \quad (3.18)$$

and its components are easily computed in the Poincaré coordinates. In particular, since  $\partial/\partial t$  is a timelike Killing vector, the energy density becomes  $\sqrt{-g}T_0^0 = \frac{z}{2}(F_+^2 + F_-^2 + F_{+-}^2) \geq 0$ .

## 4 Topologically Massive AdS<sub>3</sub> Gravitons

We now come to topologically massive gravity with a cosmological constant. Whereas in the preceding cases the cosmological term provided the desired AdS<sub>3</sub> background, here it also contributes to the dynamics through the quadratic expansion of  $\sqrt{-\det \gamma}$  about the  $g_{\mu\nu}$  vacuum. The full action for topologically massive gravity is

$$I = \int d^3x \left\{ -\sqrt{-\gamma} (R - 2\Lambda) + \frac{1}{2\mu} \varepsilon^{\mu\nu\rho} \left( \Gamma_{\mu\beta}^{\alpha} \partial_{\nu} \Gamma_{\rho\alpha}^{\beta} + \frac{2}{3} \Gamma_{\mu\gamma}^{\alpha} \Gamma_{\nu\beta}^{\gamma} \Gamma_{\rho\alpha}^{\beta} \right) \right\}. \quad (4.1)$$

Note that we have chosen the correct, “wrong” sign for the  $d = 3$  Einstein–Hilbert part; as previously explained, this was necessary in the  $\Lambda = 0$  limit for positivity of energy, and remains so here.

We linearize the dynamical metric about an AdS<sub>3</sub> background by writing

$$ds^2 = \frac{2dx^+dx^- + dz^2}{z^2} + (dx^+)^2 h_{++} + 2dx^+dz h_+ + dz^2 h + O(h^2). \quad (4.2)$$

We have chosen the light-front gauge for linearized diffeomorphisms to remove any component of the metric fluctuations  $h_{\mu\nu}$  with an  $x^-$  index. Had we not done so, our computations would ultimately have depended only on the combination

$$\varphi \equiv h - 2([\partial + \frac{1}{z}]/\partial_-)h_{-z} + ([\partial + \frac{1}{z}][\partial + \frac{2}{z}]/\partial_-^2)h_{--}, \quad (4.3)$$

which is gauge invariant under linearized diffeomorphisms

$$\delta h_{\mu\nu} = D_{\mu}\xi_{\nu} + D_{\nu}\xi_{\mu}. \quad (4.4)$$

A somewhat lengthy computation yields the terms in (4.1) quadratic in the fluctuations:

$$I = \int d\tau \left\{ -\frac{z^4}{\mu} \langle X, \dot{h} \rangle - \frac{z}{2} \left( [zX - h]^2 + \frac{z^3}{\mu} \left[ X + \partial h + \frac{\mu+2}{z} h \right] Y \right) dx^- dz \right\}$$

with  $X \equiv \partial_- h_+$ ,  $Y \equiv \partial_-^2 h_{++}$ . (4.5)

The fields  $X$  and  $Y$  appear only algebraically, so can be directly integrated out. Notice in particular that their quadratic form  $X^2 + zXY/\mu$  is degenerate only in the pure gravity limit  $\mu \rightarrow \infty$ , where the Chern–Simons term disappears. This algebraic integration leaves the simple “scalar” action\*

$$I = \int d\tau \left\{ z^3 \langle h, \dot{h} \rangle - \frac{z}{2} ([z\partial + \mu + 3]h)^2 dx^- dz \right\}. \quad (4.6)$$

Had we not chosen light-front gauge, but simply integrated out the auxiliary fields  $X$  and  $Y$ , we would have found the same result, but with  $h$  replaced by the gauge invariant combination  $\varphi$  defined in (4.3). The light-front gauge choice is then simply a field redefinition.

The field rescaling

$$\phi = z^{3/2} h \quad (4.7)$$

now yields our standard scalar action

$$I = \int d\tau \left\{ \langle \phi, \dot{\phi} \rangle - \left( \frac{1}{2} [\partial \phi]^2 + \frac{1}{2z^2} \left[ (\mu+2)^2 - \frac{1}{4} \right] \phi^2 \right) dx^- dz \right\}, \quad (4.8)$$

with the mass  $m$  given by

$$m^2 = (\mu+2)^2 - 1. \quad (4.9)$$

Again, half-integer values of  $\mu$  lead to lightlike propagation, with the Breitenlohner–Freedman bound saturated at  $\mu = -2$ . Comparing (3.7) and (4.9),

---

\*Given the analysis in [6], one might wonder whether integrating out the physical field  $h$  rather than the “ghost”  $\partial_- h_+$  could, at least for special values of  $\mu$ , lead to an interesting “dual” result. This is not the case. Integrating out  $h$  leads to a second order action in  $h_+$ ; if this action is rewritten in first order form, using a canonical momentum  $\pi_+$  for  $\partial_- h_+$ , the field  $h_{++}$  becomes a Lagrange multiplier for a constraint that eliminates the ghost  $h_+$ . The rescaling  $\pi_+ = z^{5/2} \phi/\mu$  then yields exactly the scalar action (4.8), with the correct physical sign.

the difference between vector and tensor masses is the simple displacement  $\mu + 1 \mapsto \mu + 2$ .

As in electrodynamics, we find an apparently chirally asymmetric mass formula. However, just as we were able to write scalar wave equations for the gauge invariant field strengths in topologically massive electrodynamics, a similar manipulation is possible here. We first observe that  $\partial_-^2 \varphi$  can be written as a gauge invariant combination

$$\partial_-^2 \varphi = \partial_-^2 h - 2\left[\partial + \frac{1}{z}\right]\partial_- h_- + \left[\partial + \frac{1}{z}\right]\left[\partial + \frac{2}{z}\right]h_{--} = -\frac{2}{z^2} \mathcal{H}_{--} \quad (4.10)$$

of the metric fluctuations, where  $\mathcal{H}_{--}$  is the chirality  $-2$  component of the linearized cosmological Einstein tensor

$$\mathcal{H}_{\rho\sigma} = [G_{\rho\sigma} - g_{\rho\sigma}]_{\text{LINEAR}}. \quad (4.11)$$

It is crucial here that  $\mathcal{H}_{\rho\sigma}$  is gauge invariant with respect to the linearized diffeomorphisms (4.4): at this order, a transformation of the form (4.4) acting on  $\mathcal{H}_{\rho\sigma}$  yields the Lie derivative of the *background* cosmological Einstein tensor  $G_{\rho\sigma} - g_{\rho\sigma}$ . This transformation therefore vanishes for an anti-de Sitter background. Hence, just as in cosmological topologically massive electrodynamics, we describe the physical excitations in terms of these gauge invariant variables.

In this language, the field equations are

$$\mathcal{H}_{\rho\sigma} - \frac{1}{\mu} \frac{1}{\sqrt{-g}} \varepsilon_{(\rho}^{\alpha\beta} D_{|\alpha|} \mathcal{H}_{\sigma)\beta} = 0, \quad (4.12)$$

along with the Bianchi identity

$$D^\rho \mathcal{H}_{\rho\sigma} = 0. \quad (4.13)$$

From these equations it follows that

$$(\Delta - \mu^2 + 3)\mathcal{H}_{\rho\sigma} = 0, \quad \mathcal{H}_\rho^\rho = 0. \quad (4.14)$$

Writing out equations (4.12–4.14) in the Poincaré frame (2.3) yields scalar wave equations for *each* gauge invariant mode  $\mathcal{H}_{\rho\sigma}$ :

$$\left[2\partial_- \partial_+ + \partial^2 - \frac{(\mu - \text{sign}_{\rho\sigma})^2 - 1/4}{z^2}\right] \left(\frac{1}{\sqrt{z}} \mathcal{H}_{\rho\sigma}\right) = 0. \quad (4.15)$$

Here  $\text{sign}_{\rho\sigma}$  denotes the sum of the indices  $\rho + \sigma$  in light-front coordinates (counting  $z$  as 0), and keeps track of the chirality shifts of the mass parameter  $\mu$ . The resulting masses for the gauge invariant modes are

Field	$m^2$
$\mathcal{H}_{++}$	$(\mu - 2)^2 - 1$
$\mathcal{H}_{+z}$	$(\mu - 1)^2 - 1$
$\mathcal{H}_{zz}, \mathcal{H}_{+-}$	$\mu^2 - 1$
$\mathcal{H}_{-z}$	$(\mu + 1)^2 - 1$
$\mathcal{H}_{--}$	$(\mu + 2)^2 - 1$

(4.16)

Note the (predicted) invariance under  $\mu \mapsto -\mu$  and a chirality flip.

At this point it is clear that linearized cosmological topologically massive gravity possesses local “scalar” field theoretic degrees of freedom for any value of  $\mu$ , and that these cannot be gauged away, since  $\mathcal{H}_{\mu\nu}$  is gauge invariant. Since the mass respects the Breitenlohner–Freedman bound, these excitations are described by a consistent, positive energy field theory. But we must still investigate whether this theory’s metric fluctuations decay asymptotically.

## 5 Asymptotics

We might next ask whether every solution for the curvature fluctuation  $\mathcal{H}_{\mu\nu}$  corresponds to a genuine fluctuating metric. The answer is yes, at least at this order. In three dimensions, a perturbation of the Einstein tensor uniquely determines a perturbation of the full curvature tensor. If  $\mathcal{H}_{\mu\nu}$  is divergence-free with respect to the background metric—as it is in our case—then the perturbed curvature tensor will automatically satisfy the linearized Bianchi identities, which are the integrability conditions for the existence of a connection and metric.

If we further require that these metric fluctuations produce a cosmological Einstein tensor that vanishes asymptotically, the situation becomes more subtle. Recall from Section 2 that the Poincaré patch contains two pieces of the anti-de Sitter boundary, a surface at  $z \rightarrow 0$  and an additional line at  $z \rightarrow \infty$ . We start by considering  $z = 0$ . Bulk solutions  $\mathcal{H}_{\rho\sigma}$  to the wave equation (4.15) depend on  $z$  through a Bessel function with index  $\nu =$

$\mu - \text{sign}_{\rho\sigma}$ , and are given explicitly in the Appendix. Taking  $\mu > 0$  (say), such solutions will die off at  $z = 0$  for large enough  $\mu$ . Indeed, when  $0 < \mu \neq 1$ , the asymptotics of the various components of the linearized cosmological Einstein tensor are

Field	Asymptotics
$\mathcal{H}_{++}$	$z^{\mu-1}$
$\mathcal{H}_{+z}$	$z^\mu$
$\mathcal{H}_{zz}, \mathcal{H}_{+-}$	$z^{\mu+1}$
$\mathcal{H}_{-z}$	$z^{\mu+2}$
$\mathcal{H}_{--}$	$z^{\mu+3}$

(5.1)

When  $\mu = 1$ , the Bessel function identity (valid for integer index)

$$J_n(x) = (-)^n J_{-n}(x) \quad (5.2)$$

implies asymptotics

Field	Asymptotics
$\mathcal{H}_{++}$	$z^2$
$\mathcal{H}_{+z}$	$z$
$\mathcal{H}_{zz}, \mathcal{H}_{+-}$	$z^2$
$\mathcal{H}_{-z}$	$z^3$
$\mathcal{H}_{--}$	$z^4$

(5.3)

where all modes decay as  $z \rightarrow 0$ . For negative  $\mu$ , the same asymptotics hold but for opposite chiralities  $+\leftrightarrow -$ . Only when  $0 < |\mu| < 1$  do components fail to decay.

Of course, it is inconclusive to examine bare components of curvatures, since the unperturbed anti-de Sitter metric is singular at  $z = 0$  in our coordinates. Indeed, contracting the curvature fluctuations in (5.1) with anti-de Sitter unit vectors gives an extra factor  $z^2$ , so no value of  $\mu$  gives dangerous asymptotics for curvatures at  $z = 0$ . Moreover, invariants such as the scalar curvature and squares of the Ricci and Riemann tensors remain well-behaved. For any  $\mu > 0$ , the perturbed metric is still asymptotically anti-de Sitter at  $z = 0$ , in the sense that  $z^2 ds^2$  extends to the conformal boundary with no singularities.

In the context of the AdS/CFT correspondence, stronger conditions than the standard [21] conformal boundary conditions of general relativity are sometimes imposed. It is common to express the bulk metric in Gaussian normal coordinates

$$ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} \left[ 2dx^+ dx^- + z^2 \left( \hat{h}_{++} (dx^+)^2 + 2\hat{h}_{+-} dx^+ dx^- + \hat{h}_{--} (dx^-)^2 \right) \right] \quad (5.4)$$

and demand that the induced boundary metric satisfy the Fefferman–Graham asymptotic conditions [14] that  $(\hat{h}_{++}, \hat{h}_{+-}, \hat{h}_{--})$  be finite at  $z \rightarrow 0$ . It is not difficult to compute the relation between the Einstein tensor fluctuations  $\mathcal{H}_{\mu\nu}$  and the Gaussian normal metric ones:

$$\begin{aligned} \mathcal{H}_{++} &= -\frac{1}{2}N(N+2)\hat{h}_{++} \\ \mathcal{H}_{+-} &= -\frac{1}{2}N(N+2)\hat{h}_{+-} \\ \mathcal{H}_{--} &= -\frac{1}{2}N(N+2)\hat{h}_{--} \end{aligned} \quad (5.5)$$

where the Euler operator

$$N \equiv z \frac{\partial}{\partial z} \quad (5.6)$$

is in fact the unit normal vector to the anti-de Sitter boundary. We can now easily find the asymptotic solutions for the Gaussian normal metric fluctuations corresponding to the curvature asymptotics in (5.1):

Field	Asymptotics	
$\hat{h}_{++}$	$z^{\mu-1}$	(5.7)
$\hat{h}_{+-}$	$z^{\mu+1}$	
$\hat{h}_{--}$	$z^{\mu+3}$	

When  $\mu > 1$ , the metric decays at least as fast as the Fefferman–Graham



asymptotic conditions. The value  $\mu = 1$  yields metric asymptotics

Field	Asymptotics
$\hat{h}_{++}$	$z^2$
$\hat{h}_{+-}$	$z^2$
$\hat{h}_{--}$	$z^4$

(5.8)

again obeying the Fefferman–Graham conditions.

The situation for large  $z$ , on the other hand, is more problematic. As noted in [22], Bessel functions behave for large  $z$  as

$$J_\nu(\omega z) \sim \sqrt{\frac{2}{\pi \omega z}} \cos\left(\omega z - \frac{(2\nu + 1)\pi}{4}\right), \quad (5.9)$$

so the curvatures  $\mathcal{H}$  given in the Appendix all diverge as  $z^{1/2}$ , independent of the mass parameter  $\mu$ . Our linear approximation thus breaks down at large  $z$ , and in particular at the portion of the anti-de Sitter boundary contained in our Poincaré patch at  $z \rightarrow \infty$ .

While the individual modes blow up for large  $z$ , however, it is important to remember that we are working in a linearized theory, in which we can form superpositions. We now show that our modes can be used to construct finite norm, finite energy wave packets with support away from large  $z$ . In particular, such wave packets exist at the critical value  $\mu = 1$ , where they describe a propagating bulk degree of freedom that satisfies anti-de Sitter boundary conditions.

## 6 Wave Packets

Superpositions of our linear solutions are most easily described in light-front gauge, in which linearized topologically massive gravity is characterized by a single unconstrained field  $h$ . The modes we found in Section 4 take the form

$$h_{\omega k}(x, z, t) = \sqrt{\frac{\omega}{4\pi E}} \frac{1}{z} J_{\mu+2}(\omega z) e^{ikx - iEt} \quad (6.1)$$

$$h_{\omega k}^*(x, z, t) = \sqrt{\frac{\omega}{4\pi E}} \frac{1}{z} J_{\mu+2}(\omega z) e^{-ikx + iEt} \quad \text{with } E = \sqrt{\omega^2 + k^2}.$$

The action (4.8) implies the existence of a conserved bilinear current

$$\mathcal{J}_{\text{LF}}^\mu(h_1, h_2) = z^4 g^{\mu\nu} (h_1 \partial_\nu h_2 - h_2 \partial_\nu h_1) \quad (6.2)$$

in light-front gauge, which, as in ordinary Klein-Gordon theory, gives rise to a time-independent inner product

$$(h_1, h_2) = -i \int_\Sigma dx dz z^3 h_1 \overleftrightarrow{\partial}_t h_2^*. \quad (6.3)$$

Using the completeness relation for Bessel functions,

$$\int_0^\infty dz z J_\nu(\omega z) J_\nu(\omega' z) = \frac{1}{\omega} \delta(\omega - \omega'), \quad (6.4)$$

it is easy to verify that

$$(h_{\omega k}, h_{\omega' k'}) = -(h_{\omega k}^*, h_{\omega' k'}^*) = \delta(\omega - \omega') \delta(k - k'), \quad (h_{\omega k}, h_{\omega' k'}^*) = 0. \quad (6.5)$$

We can now form an arbitrary superposition

$$h(x, z, t) = \int d\omega dk [a(\omega, k) h_{\omega k}(x, z, t) + a^*(\omega, k) h_{\omega k}^*(x, z, t)], \quad (6.6)$$

which will again be a solution of the linearized field equations. Indeed, we can take an *arbitrary* profile  $\psi(x, z)$ ,  $\partial_t \psi(x, z)$  at  $t = 0$ , determine the coefficients  $a$  and  $a^*$  by

$$a(\omega, k) = (\psi, h_{\omega k}), \quad a^*(\omega, k) = -(\psi, h_{\omega k}^*), \quad (6.7)$$

and use (6.6) to give the future evolution of the field. In particular, we can choose  $\psi$  to have its support away from large  $z$ , thus avoiding the asymptotic difficulties discussed at the end of Section 5. By linearity of the field equations, the remaining light-front components  $h_+$  and  $h_{++}$  of the metric will all have essentially the same profile, and in particular will vanish anywhere  $h$  vanishes.

The norm of  $h$  is easily computed to be

$$(h, h) = \int d\omega dk [a(\omega, k)^2 - a^*(\omega, k)^2], \quad (6.8)$$

and is preserved by time evolution. As in ordinary Klein-Gordon theory, the norm is not positive definite, but, again as in ordinary Klein-Gordon

theory, one can treat the positive- and negative-frequency modes separately to obtain a positive norm.

Although the action (4.8) is simply a gauge-fixed version of the full action of cosmological topologically massive gravity, one might still worry that the inner product (6.3) may be missing some of the “gravitational” features of the theory. This is not the case. In fact, the conserved current (6.2), and therefore the inner product, is precisely the gauge-fixed version of the full symplectic current of topologically massive gravity, up to a total derivative that gives no contribution for our wave packets.

More explicitly, any covariant Lagrangian  $L$ , depending on arbitrary fields  $\varphi$ , gives rise to a conserved symplectic current  $\mathcal{J}$  on covariant phase space through the prescription [23, 24]

$$\begin{aligned}\delta_1 L[\varphi] &= E[\varphi] \delta_1 \varphi + \nabla_\mu \Theta^\mu[\varphi, \delta_1 \varphi], \\ \mathcal{J}^\mu[\varphi, \delta_1 \varphi, \delta_2 \varphi] &= \delta_1 \Theta^\mu[\varphi, \delta_2 \varphi] - \delta_2 \Theta^\mu[\varphi, \delta_1 \varphi],\end{aligned}\tag{6.9}$$

where the equations of motion are  $E = 0$  and  $\delta_{1,2}\varphi$  satisfy the linearized field equations. For cosmological topologically massive gravity, this current is

$$\begin{aligned}\sqrt{-g} \mathcal{J}_{\text{TMG}}^\rho[g, \delta_1 g, \delta_2 g] &= \delta_1(\sqrt{-g} g^{\mu\nu}) \delta_2 \Gamma_{\mu\nu}^\rho - \delta_1(\sqrt{-g} g^{\rho\mu}) \delta_2 \Gamma_{\mu\sigma}^\sigma \\ &\quad + \frac{1}{2\mu} (\epsilon^{\mu\rho\sigma} \delta_1 \Gamma_{\mu\beta}^\alpha \delta_2 \Gamma_{\sigma\alpha}^\beta - 2\epsilon^{\rho\alpha\gamma} \delta_2 g_{\mu\alpha} \mathcal{H}^\mu{}_\gamma[\delta_1 g]) - (1 \leftrightarrow 2).\end{aligned}\tag{6.10}$$

A tedious but straightforward computation then shows that in light-front gauge,

$$\mathcal{J}_{\text{TMG}}^\rho[g, \delta_1 g, \delta_2 g] = \mathcal{J}_{\text{LF}}^\rho[g, h_1, h_2] + \nabla_\mu \mathcal{F}^{[\mu\rho]},\tag{6.11}$$

where the superpotential  $\mathcal{F}^{[\mu\rho]}$  makes no contribution to the inner product as long as the wave packets fall off fast enough at the boundary.\*

We can next evaluate the energy of our wave packets. We start again in the light-front formalism. We show in the Appendix that the conserved energy takes the form

$$H = \frac{1}{2} \int dx dz z^3 \left[ (\partial_t h)^2 + (\partial_x h)^2 + (\partial_z h)^2 + \frac{(\mu + 2)^2 - 1}{z^2} h^2 \right],\tag{6.12}$$

---

\*The computation is relatively easy for the  $+$  and  $z$  components of  $\mathcal{J}$ ; one can avoid the considerably more complicated calculation of the  $-$  component by integrating the conservation equation.

where we have used equations (2.7) and (4.7) to translate from  $\varphi$  to  $h$  and (4.9) to rewrite the mass in terms of  $\mu$ . To evaluate this expression, one can use the equations of motion (2.8) and integrate by parts—again assuming that our wave packets fall off fast enough at the boundary—to obtain the simple expression

$$H = \frac{1}{2} \int dx dz z^3 [h \partial_t^2 h - (\partial_t h)^2] . \quad (6.13)$$

The superposition (6.6) then gives the elegant result that

$$H = \int d\omega dk \sqrt{k^2 + \omega^2} |a(\omega, k)|^2 , \quad (6.14)$$

which is clearly positive, and finite for suitable choices of coefficients  $a(\omega, k)$ .

This expression for energy was obtained from the gauge-fixed action, and one might again worry that it misses some “gravitational” features. Again, it does not. For an arbitrary covariant theory, the symplectic current (6.9) determines the symplectic structure on the covariant phase space. Let  $\Sigma$  be any spacelike surface, with induced metric  $\tilde{g}_{\mu\nu}$  and unit normal  $n^\mu$ . Then Hamilton’s equations of motion require that the Hamiltonian  $H$  corresponding to evolution by a Killing vector  $\xi^\mu$  (with Lie derivative  $\mathcal{L}_\xi$ ) must satisfy

$$\delta H = \int_\Sigma d^{n-1}x \sqrt{\tilde{g}} n_\mu \mathcal{J}^\mu[\varphi, \delta\varphi, \mathcal{L}_\xi\varphi] \quad (6.15)$$

for any variation  $\delta\varphi$  of the fields [24]. In particular, if  $\delta\varphi$  is a small fluctuation around a background, the expression (6.15) gives the contribution of that fluctuation to the total energy. But we have already seen that the full symplectic current of topologically massive gravity is equivalent to the light-front current (6.2). Substituting  $h_2 = \mathcal{L}_t h_1$  and integrating over a constant time surface, we recover equation (6.13) for the energy.

We can now return to the question of asymptotics. In the previous Section, we showed that individual modes obeyed Fefferman–Graham asymptotics at the  $z = 0$  boundary, and the preceding analysis shows that we can form wavepackets from these modes with support only at the  $z = 0$  boundary on any time slice. However, one may worry that these wavepackets will propagate to the boundary at  $z = \infty$  and produce non-asymptotically anti-de Sitter metric fluctuations there. This is not the case, as we shall now show—finite energy wavepackets always correspond to asymptotically anti-de Sitter

metric fluctuations. Let us assume  $\mu > 0$ —the case  $\mu < 0$  is equivalent under a flip of chirality—and choose an initial profile for  $h$  that has finite energy. The energy remains constant as the wave packet evolves, and since each term in the integrand of (6.12) is nonnegative, none can diverge. Near the boundary at  $z = 0$ , finiteness requires that  $h \sim z^{-1+\varepsilon_1}$  and  $\partial_\mu h \sim z^{-2+\varepsilon_2}$  with  $\varepsilon_1, \varepsilon_2 > 0$ . In turn, the alternate energy formula (6.13) and the equations of motion for  $h$  imply that  $\partial_\mu \partial_\nu h \sim z^{-3+\varepsilon_3}$  ( $\varepsilon_3 > 0$ ). From (4.10) and (A.27), it follows that the components  $\mathcal{H}_{\rho\sigma}$  of the curvature fluctuations go as  $z^{-1+\delta}$  with  $\delta > 0$ , and that the invariant components, obtained by contracting with anti-de Sitter unit vectors, fall off as  $z^{1+\delta}$ . Moreover, from eqn. (5.5), the components  $\hat{h}_{\rho\sigma}$  of the metric fluctuations in Gaussian normal coordinates must also go as  $z^{-1+\delta}$ . While this is not quite strong enough to guarantee Fefferman-Graham asymptotics, it is sufficient to ensure that the spacetime remains asymptotically anti-de Sitter near  $z = 0$  for all times.

We next turn to the portion of the boundary at  $z \rightarrow \infty$ . The coordinates of our initial Poincaré patch are not well suited to describing this region, but we can perform the inversion  $x^\mu = -\tilde{x}^\mu / (\tilde{x}^\nu \eta_{\nu\rho} \tilde{x}^\rho)$  discussed in Section 2 to obtain new coordinates for which this boundary occurs at  $\tilde{z} = 0$ . This transformation takes us out of the light front frame, and a further infinitesimal transformation is needed to restore light front coordinates for the metric fluctuations  $h$ . Once this is done, though, the energy is again of the form (6.12), with  $x^\mu$  replaced by  $\tilde{x}^\mu$ ; and since  $H$  is coordinate-independent, its value is again finite (and, indeed, identical to our initial value). By exactly the same arguments that led to asymptotically anti-de Sitter behavior at  $z = 0$ , the metric must be asymptotically anti-de Sitter at  $\tilde{z} = 0$ . More generally, we can choose Poincaré coordinates near any portion of the boundary, and finiteness of the energy will always require asymptotically anti-de Sitter behavior of the excitations.

None of these considerations depend on the choice of  $\mu$ . In particular, well-behaved, finite energy, propagating wave packets occur at the critical value  $\mu = 1$ . We stress again that these modes cannot be gauged away: they are expressed in terms of the gauge invariant linearized cosmological Einstein tensor. To be sure, these propagating configurations are not eigenfunctions of the  $\text{SL}(2, \mathbb{R})$  generator  $L_0$  of [6], so this does not *directly* contradict their results. But the suggestion of [6] that the theory should have no bulk modes at  $\mu = 1$  apparently misses the modes studied here.

Although the  $\mu = 1$  theory has bulk modes, it does have several distinguishing features, which we take up in the following sections.

## 7 Chern–Simons and Chiral Gravity

Chern–Simons formulations have proved quite useful in understanding gravity and supergravity in  $2 + 1$  dimensions [17, 18, 25–28]. In fact, one might adopt the motto that *all* three-dimensional theories of gravity are Chern–Simons. Let us see how this is borne out for cosmological topologically massive gravity. Define a pair of connections

$${}^{\pm}A_{\mu}{}^a{}_b = \omega_{\mu}{}^a{}_b \pm \varepsilon^a{}_{bc} e_{\mu}{}^c, \quad (7.1)$$

where  $\omega$  and  $e$  are the spin connection and dreibein. The associated Chern–Simons actions are

$$I[A] = \frac{1}{2} \int d^3x \, \varepsilon^{\mu\nu\rho} \left( A_{\mu}{}^a{}_b \partial_{\nu} A_{\rho}{}^b{}_a + \frac{2}{3} A_{\mu}{}^a{}_c A_{\nu}{}^c{}_b A_{\rho}{}^b{}_a \right). \quad (7.2)$$

If we impose the torsion-free condition as a constraint [29, 30]  $\omega = \omega[e]$ , these actions become

$$I[{}^{\pm}A[e]] = \int d^3x \left\{ \pm \sqrt{-\gamma} (R[\omega] - 2\Lambda) + \frac{1}{2} \varepsilon^{\mu\nu\rho} \left( \omega_{\mu}{}^a{}_b \partial_{\nu} \omega_{\rho}{}^b{}_a + \frac{2}{3} \omega_{\mu}{}^a{}_c \omega_{\nu}{}^c{}_b \omega_{\rho}{}^b{}_a \right) \right\}. \quad (7.3)$$

Hence the complete cosmological topologically massive gravity action (4.1) may be written as

$$I_{\text{TMG}}[e] = -\frac{1}{2} \left(1 - \frac{1}{\mu}\right) I[{}^{+}A[e]] + \frac{1}{2} \left(1 + \frac{1}{\mu}\right) I[{}^{-}A[e]]. \quad (7.4)$$

The coefficients correspond to the central charges of left and right components [4, 5].

At the critical values of the topological mass parameter,  $\mu = \pm 1$ , the action reduces to

$$I[e] = -I_{\text{TMG}}[e] \Big|_{\mu=\pm 1} = \pm I[{}^{\pm}A[e]]. \quad (7.5)$$

The third order action (7.4) is reminiscent of the second order Achúcarro–Townsend–Witten [17, 18] formulation of ordinary  $d = 3$  cosmological Einstein gravity as a sum of left and right  $\text{SL}(2, \mathbb{R})$  Chern–Simons terms,

$$I_{\text{E}+\Lambda}[e] = I[{}^{+}A[e]] - I[{}^{-}A[e]]. \quad (7.6)$$

The model of (7.5) corresponds simply to discarding one of the two terms.\* The theory is *not* topological; it still has propagating bulk modes that arise because of the dependence of the connection  $A[e]$  on the derivatives of the dreibein [31]. Amusingly (7.4) has an exact counterpart in the vector model [32].

One can also investigate the presence of bulk modes beyond linear perturbation theory by examining the structure of the constraints of the full theory based on this reformulation. The result [33, 34] is that the canonical analysis of [29] for the case of vanishing cosmological constant generalizes in a straightforward manner, with no jump in the number or nature of the constraints or change in the number of degrees of freedom at the critical value of  $\mu$ .

## 8 Critical Massive Gravitons as Photons

We next turn to an intriguing relationship between gravity and electrodynamics, and show that linearized cosmological topologically massive gravity at the critical point  $\mu = 1$  reduces to topologically massive electrodynamics in the same anti-de Sitter background. We work in terms of the linearized cosmological Einstein tensor  $\mathcal{H}_{\mu\nu}$  defined in equation (4.11), and begin by observing that the equations of motion (4.13) and (4.14) evaluated at a critical point  $\mu^2 = 1$ ,

$$(\Delta + 2)\mathcal{H}_{\mu\nu} = 0 = D^\mu \mathcal{H}_{\mu\nu} = \mathcal{H}_\mu^\mu, \quad (8.1)$$

are exactly those obeyed by the metric fluctuations of pure three-dimensional cosmological Einstein gravity in harmonic gauge. But pure three-dimensional gravity has no field theoretic degrees of freedom—all its fluctuations are “pure gauge.” We can thus conclude that

$$\mathcal{H}_{\mu\nu} = D_{(\mu} \tilde{F}_{\nu)}, \quad (8.2)$$

where  $\tilde{F}_\nu$  is some (suggestively labeled) vector field.

Our aim is therefore to reformulate linearized chiral cosmological topologically massive gravity in terms of the vector field  $\tilde{F}_\nu$ . We first write out the implications of the equation of motion (4.12) for  $\tilde{F}_\nu$ :

$$D_{(\mu} \left( \tilde{F}_{\nu)} - \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_{\nu)}^{\alpha\beta} D_\alpha \tilde{F}_\beta \right) = 0. \quad (8.3)$$

---

\*R. Jackiw and D. Grumiller (private communication) have also noted this.

This equations says that the quantity  $\tilde{F}_\nu - \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_\nu^{\alpha\beta} D_\alpha \tilde{F}_\beta$  vanishes modulo a Killing vector  $k_\nu$ , *i.e.*,

$$\tilde{F}_\nu - \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_\nu^{\alpha\beta} D_\alpha \tilde{F}_\beta = k_\nu, \quad D_{(\mu} k_{\nu)} = 0. \quad (8.4)$$

At this juncture we know that any  $\mathcal{H}_{\mu\nu}$  that obeys (8.1) satisfies the ansatz (8.2), with  $\tilde{F}_\nu$  solving (8.4) for some Killing vector  $k_\nu$ . If we want to take  $\tilde{F}_\nu$  as our independent field, however, we must ask the converse question: which such  $\tilde{F}_\nu$  lead to linearized field strengths obeying (8.1)?

To answer this, we start with (8.4), and note that its divergence implies

$$D^\mu \tilde{F}_\mu = 0. \quad (8.5)$$

This guarantees that  $\mathcal{H}_\mu^\mu = 0$ .

We next note that (8.4) can be schematically written as  $(\mathbb{1} - \text{curl})\tilde{F}_\nu = 0$ ; acting with the operator  $\mathbb{1} + \text{curl}$  yields

$$-\frac{1}{4}(\Delta - 2)\tilde{F}_\nu = k_\nu + \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_\nu^{\alpha\beta} D_\alpha k_\beta. \quad (8.6)$$

The divergence-free condition  $D^\mu \mathcal{H}_{\mu\nu} = 0$ , on the other hand, requires that  $(\Delta - 2)\tilde{F}_\nu = 0$ , which is also sufficient to enforce the final field equation  $(\Delta + 2)\mathcal{H}_{\mu\nu} = 0$ . We must therefore restrict (8.4) to Killing vectors for which the right-hand side of (8.6) vanishes,

$$k_\nu + \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_\nu^{\alpha\beta} D_\alpha k_\beta = 0. \quad (8.7)$$

We thus see that equations (8.4) and (8.7) together imply the full set of field equations of linearized cosmological topologically massive gravity. The Killing vector  $k_\nu$  is irrelevant: we can use (8.7) to shift  $\tilde{F}_\nu$  in (8.4), rewriting it in the form

$$\tilde{F}_\nu + k_\nu - \frac{1}{2} \frac{1}{\sqrt{-g}} \varepsilon_\nu^{\alpha\beta} D_\alpha (\tilde{F}_\beta + k_\beta) = 0. \quad (8.8)$$

Since the linearized field strength in (8.2) depends only on the symmetric derivative of  $\tilde{F}_\nu$ , any shift by a Killing vector drops out.

Finally, calling

$$F_{\mu\nu} = \frac{1}{\sqrt{-g}} \varepsilon_{\mu\nu\rho} (\tilde{F}^\rho + k^\rho), \quad (8.9)$$



we can write our equations in the form

$$D^\mu F_{\mu\nu} - \frac{1}{\sqrt{-g}} \varepsilon_\nu{}^{\mu\rho} F_{\mu\rho} = 0 = D_{[\mu} F_{\nu\rho]}. \quad (8.10)$$

These are precisely the field equations for cosmological topologically massive electrodynamics with a mass parameter  $\mu_{\text{EM}} = 2$ . (This value is commensurate with the maximal chirality electromagnetic field strength being +1, rather than +2 for gravitons.)

Our recipe is thus to solve the field equations of topologically massive electrodynamics in an anti-de Sitter background for the electromagnetic field strength  $F_{\mu\nu}$  or its dual  $\tilde{F}_\nu$ . Such a vector determines a gauge invariant solution to linearized cosmological topologically massive gravity. Conversely, up to questions of topology and boundary conditions, any linearized solution of the gravitational theory can be obtained in this manner. In other words, we have found a simple correspondence (8.2) between the single degree of freedom for topologically massive spin 2 and spin 1 modes at  $\mu = 1$ .

## 9 The Chiral Spectrum

As already noted in Section 2, our general bulk graviton solutions can be supplemented with additional chiral solutions of the field equations. To study these, we start with a representation theoretic approach. The  $\text{AdS}_3$  background has  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  isometry group with Killing vectors

$$\begin{aligned} L_+ &= \partial_+, & R_+ &= \partial_-, \\ L_0 &= -x^+ \partial_+ - \frac{1}{2} z \partial, & R_0 &= -x^- \partial_- - \frac{1}{2} z \partial, \\ L_- &= -x^+ (x^+ \partial_+ + z \partial) + \frac{1}{2} z^2 \partial_-, & R_- &= -x^- (x^- \partial_- + z \partial) + \frac{1}{2} z^2 \partial_+, \end{aligned} \quad (9.1)$$

which obey the commutation relations

$$\begin{aligned} [L_0, L_\pm] &= \pm L_\pm, & [R_0, R_\pm] &= \pm R_\pm, \\ [L_+, L_-] &= 2L_0, & [R_+, R_-] &= 2R_0. \end{aligned} \quad (9.2)$$

The left and right Casimirs  $\Delta_L = \{L_+, L_-\} + 2L_0^2$  and  $\Delta_R = \{R_+, R_-\} + 2R_0^2$  are equal, and their sum yields the invariant scalar Laplacian

$$\Delta = \Delta_L + \Delta_R = z^2 \left( 2\partial_+ \partial_- + z \partial \frac{1}{z} \partial \right). \quad (9.3)$$

In the Appendix, we show that the bulk solutions we have discussed so far are not chiral (in the three-dimensional sense), and review the result that the bulk-boundary propagator is the intertwiner between the reducible representation of the isometry group generated by the Killing vectors and irreducible quasiprimary boundary fields. There are, however, additional chiral solutions, obtained by requiring that  $L_+$  and  $R_+$  annihilate the highest chirality field  $\mathcal{H}_{--}$  and that  $R_+$  annihilate all other curvature fluctuations. In this case one finds discrete series representations for one of the  $\text{SL}(2, \mathbb{R})$  factors and a singlet for the other, and therefore a chiral subsector of the theory.

This phenomenon holds at any value of  $\mu$ , but is most easily exhibited at the critical value  $\mu = 1$ , where we can use the gravity/electromagnetism duality of Section 8. It is easy to verify that the field strength

$$F = z^{\mu-1} f(x^+) dx^+ \wedge dz = -d\left(\frac{z^\mu}{\mu} f(x^+) dx^+\right), \quad (9.4)$$

obeys the topologically massive Maxwell's equations in an  $\text{AdS}_3$  background. Applying the duality relations (8.2) and (8.9) at  $\mu_{\text{EM}} = 2$ , we find the curvature fluctuations

$$\mathcal{H}_{++} = z^2 f'(x^+), \quad \mathcal{H}_{+z} = 2z f(x^+). \quad (9.5)$$

Then using (5.5) to compute the metric fluctuations, we obtain the metric

$$ds^2 = \frac{2dx^+(dx^- - \frac{1}{8}z^2 f'(x^+) dx^+) + dz^2}{z^2}. \quad (9.6)$$

This is an  $\text{AdS}_3$  pp-wave, and in fact solves the cosmological topologically massive field equations exactly, by virtue of having a vanishing cosmological Einstein tensor.

We can also construct similar solutions at arbitrary values of  $\mu$ . To that end, we consider the ansatz

$$h_{++} = z^\gamma h(x^+), \quad (9.7)$$

and compute the full equations of motion for cosmological topologically massive gravity. We find only a single nonvanishing component,

$$G_{++} - g_{++} - \frac{1}{\mu\sqrt{-g}} \varepsilon_{(+}{}^{\alpha\beta} D_{|\alpha|} G_{+)\beta} = \frac{z^\gamma h(x^+)}{2\mu} \gamma(\gamma+2)(\gamma-\mu+1) = 0. \quad (9.8)$$

Three solutions exist:  $\gamma = -2$ ,  $0$ , or  $\mu - 1$ . The first two are also pure Einstein gravity solutions; of these, only  $\gamma = 0$  gives well-behaved Fefferman–Graham asymptotics. The third is a chiral topologically massive pp-wave, the generalization of (9.6) to arbitrary  $\mu$ ; it differs from the bulk solutions we have discussed in the previous sections. As noted in [20], at the limit  $\mu \rightarrow 1$  a new logarithmic solution appears. We also see again that  $\mu = 1$  is the lowest value at which the Fefferman–Graham asymptotics are finite at the boundary, although, as stressed in [20], the logarithmic solution is still asymptotically anti-de Sitter.

## 10 Conclusions

We have analyzed cosmological topologically massive gravity, the combination of the Einstein and Chern–Simons actions in  $d = 3$  with a nonvanishing cosmological constant. With a “wrong” sign Einstein term, necessary in the  $\Lambda = 0$  limit, the theory contains a single, positive energy, massive spin-2 excitation, which can be described by a “scalar.” Indeed, the gauge invariant Einstein tensor components themselves obey scalar wave equations with masses that depend on their chirality. All components are related to one another by Bianchi identities and the field equations, and describe a single degree of freedom. This unexpected phenomenon is reminiscent of the very different, massive *non*-gauge invariant Pauli-Fierz field in anti-de Sitter, where the cosmological background breaks the degeneracy of Minkowski states [10, 11]. Cosmological topologically massive electrodynamics behaved analogously.

With our choice of coordinates, individual Bessel function modes diverge at large values of  $z$ . We have shown, however, that well-behaved, finite energy wave packets exist for all values of the couplings. These allow us to define scattering states: we can choose a complete set of wave packets near the anti-de Sitter boundary at an early time, allow them to evolve, and project them against another complete set of wave packets near the boundary at a later time, without violating the requirement that the metric be asymptotically anti-de Sitter.

We have also examined the particular,  $\mu = 1$ , version of cosmological topologically massive gravity with nonstandard sign of  $G$  suggested in [6]. This sign choice leads to negative energy for bulk modes, which unfortunately persist for all values of  $\mu$ . Nonetheless, we have found interesting new results

at  $\mu = 1$ . In particular, there is an intriguing equivalence between linearized gravity at its critical coupling and topologically massive electrodynamics, and a reduction from two to a single term in our pure Chern–Simons formulation.

It has recently been stressed in [15] that the theory is chiral at  $\mu = 1$  in the sense that the boundary conformal field theory is chiral. Indeed, we know from [4,5] that one of the boundary central charges vanishes at this point, and it was independently shown in [33] that the corresponding boundary term in the generator of diffeomorphisms is also zero. Our results do not contradict this “chirality conjecture,” which merely implies that our bulk modes should have a vanishing “charge”  $E \pm J$  at the boundary. In particular, the extended gauge invariance discussed in [15] is still diffeomorphism invariance, albeit with one chiral sector extended all the way out to the boundary. The corresponding transformations cannot remove our bulk modes, which have nonvanishing diffeomorphism-invariant curvature fluctuations and nontrivial curvature invariants in the bulk.

It would be interesting to investigate the boundary charges of our bulk modes further, but the problem is a subtle one. The BTZ black hole exemplifies the pitfalls: the BTZ metric is identical for all values of  $\mu$ , and the shift in the boundary Virasoro charges cannot be seen locally, but only by considering the global diffeomorphism generators. It would also be interesting to understand how our bulk modes relate to the modes of [35] in global coordinates, and in particular whether they are equivalent.

Our results are a further illustration of the emergent rule: adding a cosmological constant to otherwise standard actions gives rise to effects unsuspected from the Minkowski perspective.

## Acknowledgments

We thank D. Grumiller, R. Jackiw, N. Johansson, W. Li, D. Marolf, W. Song, and A. Strominger for discussions. This work was supported by the National Science Foundation under grants PHY07-57190 and DMS-0636297 and by the Department of Energy under grants DE-FG02-91ER40674 and DE-FG02-92-ER40701.

*Note added in proof.* Since this paper was prepared, the literature on this subject has grown considerably; see, e.g. [37], where further references may also be found.

## Appendix Bulk Solutions and Stability

In this appendix we compute explicit bulk solutions for the topologically massive theories and examine their energies and the question of stability. We first return to the massive scalar wave equation,

$$\left[2\partial_-\partial_+ + \frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{\nu^2}{z^2}\right]\left(\frac{\varphi}{z}\right) = 0, \quad (\text{A.1})$$

where the parameter  $\nu$  is related to the standard scalar mass  $m$  by

$$\nu^2 = m^2 - 1.$$

As was shown in Sections 3 and 4, the actions for topologically massive electrodynamics and linearized topologically massive gravity can be reduced, through nonlocal field redefinitions, to the scalar action, so solutions and perturbative stability results for (A.1) carry over to our main settings.

The coordinates

$$t = \frac{x^+ - x^-}{\sqrt{2}} \quad \text{and} \quad x = \frac{x^+ + x^-}{\sqrt{2}} \quad (\text{A.2})$$

determine timelike and spacelike background Killing vectors  $\partial/\partial t$  and  $\partial/\partial x$ . We diagonalize them both by expanding in Fourier modes:

$$\varphi = \int d\omega dE f_\omega(z) a^\dagger(k, E) \exp(i[kx - Et]) + \text{h.c.}, \quad \omega \equiv \sqrt{E^2 - k^2}. \quad (\text{A.3})$$

The wave equation (A.1) implies that

$$f_\omega(z) = \frac{z J_\nu(\omega z)}{\omega^\nu}. \quad (\text{A.4})$$

We assume  $\nu > 0$ , which forces us to choose the Bessel function of the first kind that vanishes in the anti-de Sitter boundary,  $z \rightarrow 0$ .

To compute the energy of this solution, we consider the covariantly conserved stress tensor

$$T_{\mu\nu} = -\partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} g_{\mu\nu} \left[ \partial_\rho \varphi g^{\rho\sigma} \partial_\sigma \varphi + m^2 \varphi^2 \right], \quad (\text{A.5})$$

omitting improvement terms that vanish on the above configurations. Let us denote the timelike Killing vector by

$$\frac{\partial}{\partial t} = \xi^\mu \partial_\mu, \quad (\text{A.6})$$

and recall that the vector density  $\sqrt{-g} \xi^\mu T_\mu{}^\nu$  obeys the conservation equation

$$\partial_\nu \left( \sqrt{-g} \xi^\mu T_\mu{}^\nu \right) = 0. \quad (\text{A.7})$$

Hence the energy

$$H = \int dx dz \sqrt{-g} \xi^\mu T_\mu{}^t, \quad (\text{A.8})$$

is conserved as long as the surface integral

$$\int dx \sqrt{-g} \xi^\mu T_\mu{}^z \Big|_{z=0} \quad (\text{A.9})$$

vanishes. An explicit computation yields

$$\sqrt{-g} \xi^\mu T_\mu{}^z \sim J_\nu(\omega z) \left( J_{\nu+1}(\omega z) - (\nu+1) J_\nu(\omega z) \right) \sim z^{2\nu} \xrightarrow{z \rightarrow 0} 0. \quad (\text{A.10})$$

We now compute the energy density for the configurations (A.3), evaluated for simplicity at  $t = 0$  and in a frame  $(E, k) = (\omega, 0)$ , and find

$$\sqrt{-g} \xi^\mu T_\mu{}^t = \frac{z}{2} \left[ \omega J_{\nu+1}(\omega z) - \frac{\nu+1}{z} J_\nu(\omega z) \right]^2 + \frac{\nu^2 - 1}{2z} [J_\nu(\omega z)]^2. \quad (\text{A.11})$$

This is a sum of positive squares whenever  $\nu^2 - 1 = m^2 > 0$ , which is the naive expectation for the positivity of the scalar energy. As pointed out by Breitenlohner and Freedman [12], by playing the two squares off against one another, one need only impose the weaker condition  $\nu^2 > 0$ . To see this in our frame, observe that the Bessel function identity

$$-\omega J_{\nu+1}(\omega z) = \left[ \partial - \frac{\nu}{z} \right] J_\nu(\omega z) \quad (\text{A.12})$$

allows us to reexpress the energy density as

$$\sqrt{-g} \xi^\mu T_\mu{}^t = \frac{z}{2} [\partial J_\nu(\omega z)]^2 + \frac{\nu^2}{2z} [J_\nu(\omega z)]^2 + \frac{1}{2} \partial [J_\nu(\omega z)]^2. \quad (\text{A.13})$$

The first two terms are manifestly positive whenever the Breitenlohner–Freedman bound  $\nu^2 = m^2 + 1 > 0$  holds, while the final term is a vanishing surface contribution. Our analysis closely tracks that of [13], save for a different choice of coordinates.

To understand the representation theoretic content of these solutions, we consider the action of the isometry generators (9.1) on them. Let us call  $\omega_{\pm} \equiv (k \mp E)/\sqrt{2}$ , so  $-2\omega_+\omega_- = \omega^2$ . Then clearly we have

$$L_+ = i\omega_+ \text{ and } R_+ = i\omega_-, \quad (\text{A.14})$$

where these operators are to be read as acting upon the Fourier coefficients  $a^\dagger(\omega_+, \omega_-)$ . It may be checked from (A.4) that the coefficients  $f_\omega(z)$  satisfy the identities

$$\left(\omega_{\pm} \frac{\partial}{\partial \omega_{\pm}} - \frac{1}{2} z \partial + \frac{\nu+1}{2}\right) f_\omega(z) = 0 = \left[\frac{\partial}{\partial \omega_{\pm}} \left(\omega_{\pm} \frac{\partial}{\partial \omega_{\pm}} - z \partial - 1\right) + \frac{1}{2} z^2 \omega_{\mp}\right] f_\omega(z), \quad (\text{A.15})$$

which imply that  $f_\omega(z)$  is highest weight. Expressing  $x^\pm$  as  $i\partial/\partial\omega^\pm$  and giving due care to operator orderings, it is not difficult to determine the actions of the other isometries on the Fourier coefficients:

$$\begin{aligned} L_0 &= \omega_+ \frac{\partial}{\partial \omega_+} - \frac{\nu-1}{2}, & R_0 &= \omega_- \frac{\partial}{\partial \omega_-} - \frac{\nu-1}{2}, \\ L_- &= i\left(\omega_+ \frac{\partial}{\partial \omega_+} - \nu + 1\right) \frac{\partial}{\partial \omega_+}, & R_- &= i\left(\omega_- \frac{\partial}{\partial \omega_-} - \nu + 1\right) \frac{\partial}{\partial \omega_-}, \end{aligned} \quad (\text{A.16})$$

The operators in (A.14) and (A.16) obey the  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$  Lie algebra (9.2) and are in fact closely related to the conformal group acting on  $\mathbb{R}^{1,1}$ . Indeed, if we make an inverse Fourier transformation

$$a^\dagger(\omega_+, \omega_-) = \int \frac{d^2 y}{(2\pi)^2} \exp(-i[\omega_+ y^+ + \omega_- y^-]) \chi(y^+, y^-), \quad (\text{A.17})$$

we find the action

$$\begin{aligned} L_+ &= \partial_+, & R_+ &= \partial_-, \\ L_0 &= -y^+ \partial_+ - \frac{\nu+1}{2}, & R_0 &= -y^- \partial_- - \frac{\nu+1}{2}, \\ L_- &= -y^+(y^+ \partial_+ + \nu + 1), & R_- &= -x^-(x^- \partial_- + \nu + 1), \end{aligned} \quad (\text{A.18})$$

on the “boundary field”  $\chi(y^+, y^-)$ . Here the  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$  isometry group acts as the two-dimensional boundary conformal group. The boundary field  $\chi$  transforms as a weight  $\nu + 1$  quasiprimary. It is not difficult to show that the relation between boundary and bulk fields implied

by the inverse Fourier transform (A.17) and the solution (A.3) is exactly the bulk-boundary propagator. In detail: represent the Bessel function as  $J_\nu(\omega z) = \frac{1}{2\pi i} (\frac{z}{2})^\nu \int_C \frac{d\tau}{\tau^{\nu+1}} \exp(\tau - \frac{1}{4}\omega^2 z^2 \tau^{-1})$  (where  $C$  is the Hankel contour). This allows the integral over Fourier modes  $\omega$  to be performed, so that  $\varphi(z, x^\pm) = \int d^2 y \Delta(x^\pm - y^\pm, z) \chi(y^\pm)$ , with the bulk-boundary propagator given in proper time representation by  $\Delta(x^\pm, z) \sim \int_C \frac{d\tau}{\tau^\nu} \exp(\frac{\tau}{z}[z^2 + 2x^+ x^-])$ . In mathematical terms, this propagator is an intertwiner between the off-shell representation (9.1) and the irreducible on-shell one (A.18) [36].

We next turn to topologically massive electromagnetism. We first repeat the observation made in the main text that the field strength  $F_{\mu\nu}$  solves the scalar wave equation componentwise, with masses subject to the Breitenlohner–Freedman bound. Hence these modes carry positive energy exactly as in the scalar case. Further, their relation to boundary quasiprimary fields is also similar to the scalar case, although one must now include spin, as discussed in [36].

We therefore start with a single bulk mode

$$F_- = i\omega_- \exp(i[\omega_+ x^+ + \omega_- x^-]) J_{\mu+1}(\omega z) + \text{h.c.}, \quad (\text{A.19})$$

which solves the electromagnetic wave equation (3.17) that is, in turn, a consequence of the topologically massive Maxwell equations (3.14–3.16) and the Bianchi identity (3.12). Although we have formally reduced the electromagnetic action and field equations to those of a scalar, the two are not, of course, *physically* equivalent: the new question we must address is the form of the remaining components of the electromagnetic field strength. These follow from the Bessel function identity

$$\left[ \partial + \frac{\mu+1}{z} \right] J_{\mu+1}(\omega z) = \omega J_\mu(\omega z). \quad (\text{A.20})$$

Equation (3.16) then says

$$\partial_- F_{+-} = \left[ \partial + \frac{\mu+1}{z} \right] F_- = i\omega_- \exp(i[\omega_+ x^+ + \omega_- x^-]) J_\mu(\omega z) + \text{h.c.}, \quad (\text{A.21})$$

so

$$F_{+-} = \exp(i[\omega_+ x^+ + \omega_- x^-]) J_\mu(\omega z) + \text{h.c.} \quad (\text{A.22})$$

Similarly, equations (3.14) and (3.12) imply

$$2\partial_- F_+ = \left[ \partial + \frac{\mu}{z} \right] F_{+-} = \omega \exp(i[\omega_+ x^+ + \omega_- x^-]) J_{\mu-1}(\omega z) + \text{h.c.}, \quad (\text{A.23})$$



whence

$$F_+ = i\omega_+ \exp(i[\omega_+ x^+ + \omega_- x^-]) J_{\mu-1}(\omega z) + \text{h.c.} \quad (\text{A.24})$$

We have assumed  $\mu$  to be positive, and chosen the Bessel function of the first kind to ensure decaying behavior\* for small  $z$ .

When the topological mass parameter takes the value  $\mu_{\text{EM}} = 2$ , we can employ our gravity/electromagnetism duality to construct graviton solutions. From equations (8.2) and (8.9) and the Maxwell equations (3.12, 3.14–3.16), we can explicitly compute the relationship between the electromagnetic field strength and the linearized cosmological Einstein tensor:

$$\begin{aligned} \mathcal{H}_{--} &= -z \partial_- F_{-z}, \\ \mathcal{H}_{-z} &= -(z\partial + 3)F_{-z}, \\ \mathcal{H}_{zz} &= -(z\partial + 2)F_{+-} = -2\mathcal{H}_{+-}, \\ \mathcal{H}_{+z} &= (z\partial + 1)F_{+z}, \\ \mathcal{H}_{++} &= z\partial_+ F_{+z}. \end{aligned} \quad (\text{A.25})$$

From the electromagnetic solution (A.19, A.21, A.24), we find

$$\begin{aligned} \mathcal{H}_{--} &= \frac{\omega_-^2}{\omega} \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_3(\omega z) + \text{h.c.}, \\ \mathcal{H}_{-z} &= -i\omega_- \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_2(\omega z) + \text{h.c.}, \\ \mathcal{H}_{zz} &= -\omega \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_1(\omega z) + \text{h.c.} = -2\mathcal{H}_{+-}, \\ \mathcal{H}_{+z} &= i\omega_+ \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_0(\omega z) + \text{h.c.}, \\ \mathcal{H}_{++} &= -\frac{\omega_+^2}{\omega} \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_1(\omega z) + \text{h.c.} \end{aligned} \quad (\text{A.26})$$

These solutions obey the asymptotics quoted in section 5. It is also easy to verify that they obey the equations of motion (4.12) and Bianchi identity (4.13) for cosmological topologically massive gravity at  $\mu = 1$ . For

---

\*Contracting with the anti-de Sitter unit vectors  $\{e^+ = z\partial_+, e^- = z\partial, N = z\partial\}$ , all components  $e^{+\mu} N^\nu F_{\mu\nu}$ ,  $e^{+\mu} e^{-\nu} F_{\mu\nu}$  and  $e^{-\mu} N^\nu F_{\mu\nu}$  of the field strength vanish at the boundary for any value of  $\mu$ .

arbitrary values of  $\mu$ , those equations read

$$\begin{aligned}
\partial_- \mathcal{H}_{-z} &= \left[ \partial + \frac{\mu+1}{z} \right] \mathcal{H}_{--}, \\
\partial_- \mathcal{H}_{zz} &= \left[ \partial + \frac{\mu}{z} \right] \mathcal{H}_{-z}, \\
2\partial_- \mathcal{H}_{+z} &= - \left[ \partial + \frac{\mu-1}{z} \right] \mathcal{H}_{zz}, \\
2\partial_- \mathcal{H}_{++} &= - \left[ \partial + \frac{\mu-2}{z} \right] \mathcal{H}_{+z}, \\
\partial_+ \mathcal{H}_{+z} &= \left[ \partial - \frac{\mu-1}{z} \right] \mathcal{H}_{++} \\
\left[ \partial - \frac{1}{z} \right] \mathcal{H}_{-z} - \frac{1}{2} \partial_- \mathcal{H}_{zz} + \partial_+ \mathcal{H}_{--} &= 0, \\
\left[ \partial - \frac{1}{z} \right] \mathcal{H}_{zz} + \partial_- \mathcal{H}_+ + \partial_+ \mathcal{H}_- &= 0, \\
\left[ \partial - \frac{1}{z} \right] \mathcal{H}_{+z} - \frac{1}{2} \partial_+ \mathcal{H}_{zz} + \partial_- \mathcal{H}_{++} &= 0,
\end{aligned} \tag{A.27}$$

where we have eliminated  $\mathcal{H}_{+-}$  in favor of  $\mathcal{H}_{zz}$  by using the on-shell trace condition  $\mathcal{H}_\mu^\mu = 0$ . The last three relations are the Bianchi identity, while the first five are the topologically massive equations of motion and Bianchi identity combined.

We can solve the above equations for bulk modes at arbitrary  $\mu$  in much the same fashion as for the electromagnetic case. We find

$$\begin{aligned}
\mathcal{H}_{--} &= \frac{\omega_-^2}{\omega} \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_{\mu+2}(\omega z) + \text{h.c.}, \\
\mathcal{H}_{-z} &= -i\omega_- \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_{\mu+1}(\omega z) + \text{h.c.}, \\
\mathcal{H}_{zz} &= -\omega \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_\mu(\omega z) + \text{h.c.} = -2\mathcal{H}_{+-}, \\
\mathcal{H}_{+z} &= i\omega_+ \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_{\mu-1}(\omega z) + \text{h.c.}, \\
\mathcal{H}_{++} &= \frac{\omega_+^2}{\omega} \exp(i[\omega_+ x^+ + \omega_- x^-]) z J_{\mu-2}(\omega z) + \text{h.c.}
\end{aligned} \tag{A.28}$$

These results yield the asymptotics quoted in Section 5, and for  $\mu = 1$ ,

the expressions (A.28) agree with those obtained via our electromagnetic correspondence.

From these curvature components, it is not difficult to invert the equations for the linearized Einstein tensor to obtain explicit metric fluctuations. The light-front gauge is a particularly simple choice for this, but other gauges can also be used. For example, in harmonic gauge

$$D_\mu h^{\mu\nu} = 0, \quad h^\mu{}_\mu = 0, \quad (\text{A.29})$$

the metric fluctuations at  $\mu = 1$  become

$$h_{zz} = -2\tilde{h}_{+-} = J_0(\omega z) e^{i(\omega_+ x^+ + \omega_- x^-)} \quad (\text{A.30})$$

$$h_{z+} = \frac{i\omega_+}{\omega} J_1(\omega z) e^{i(\omega_+ x^+ + \omega_- x^-)} \quad (\text{A.31})$$

$$h_{z-} = \frac{i\omega_-}{\omega} \left( J_1(\omega z) + \frac{2}{\omega z} J_0(\omega z) \right) e^{i(\omega_+ x^+ + \omega_- x^-)} \quad (\text{A.32})$$

$$h_{++} = \frac{\omega_+^2}{\omega^2} \left( J_0(\omega z) - \frac{4}{\omega z} J_1(\omega z) \right) e^{i(\omega_+ x^+ + \omega_- x^-)} \quad (\text{A.33})$$

$$h_{--} = \frac{\omega_-^2}{\omega^2} \left[ \left( 1 - \frac{8}{\omega^2 z^2} \right) J_0(\omega z) - \frac{8}{\omega z} J_1(\omega z) \right] e^{i(\omega_+ x^+ + \omega_- x^-)} \quad (\text{A.34})$$

## References

- [1] S. Carlip, S. Deser, A. Waldron, D. K. Wise, “Topologically Massive AdS Gravity,” *Phys. Lett. B* **666** 272-276 (2008), arXiv:0807.0486 [hep-th].
- [2] S. Deser, R. Jackiw and S. Templeton, “Three-dimensional massive gauge theories,” *Phys. Rev. Lett.* **48**, 975 (1982); “Topologically massive gauge theories,” *Annals Phys.* **140**, 372 (1982).
- [3] S. Deser and B. Tekin, “Energy in topologically massive gravity,” *Class. Quant. Grav.* **20**, L259 (2003); S. Deser and B. Tekin, “Massive, topologically massive, models,” *Class. Quant. Grav.* **19**, L97 (2002), arXiv:hep-th/0203273.
- [4] P. Kraus and F. Larsen, “Holographic gravitational anomalies,” *JHEP* **0601**, 022 (2006), arXiv:hep-th/0508218.
- [5] S. N. Solodukhin, “Holography with gravitational Chern-Simons term,” *Phys. Rev.* **D74**, 024015 (2006), arXiv:hep-th/0509148.

- [6] W. Li, W. Song, and A. Strominger, “Chiral gravity in three dimensions,” JHEP **0804** 082 (2008), arXiv:0801.4566 [hep-th].
- [7] S. Deser, “Cosmological topological supergravity,” in *Quantum Theory of Gravity*, edited by S. M. Christensen, Adam Hilger, London, 1984; “Topologically massive gravity has positive energy” (in preparation).
- [8] L. F. Abbott and S. Deser, “Stability Of Gravity With A Cosmological Constant,” Nucl. Phys. B **195**, 76 (1982).
- [9] S. Deser, “Massive spin 3/2 theories in three-dimensions,” Phys. Lett. B **140**, 321 (1984).
- [10] S. Deser and R. I. Nepomechie, “Anomalous propagation of gauge fields in conformally flat spaces,” Phys. Lett. B **132**, 321 (1983); “Gauge invariance versus masslessness in de Sitter space,” Annals Phys. **154**, 396 (1984).
- [11] S. Deser and A. Waldron, “Gauge invariances and phases of massive higher spins in (A)dS,” Phys. Rev. Lett. **87**, 031601 (2001), arXiv:hep-th/0102166; “Partial masslessness of higher spins in (A)dS,” Nucl. Phys. B **607**, 577 (2001), arXiv:hep-th/0103198; “Stability of massive cosmological gravitons,” Phys. Lett. B **508**, 347 (2001). arXiv:hep-th/0103255; “Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations,” Phys. Lett. B **513**, 137 (2001), arXiv:hep-th/0105181.
- [12] P. Breitenlohner and D. Z. Freedman, “Stability in gauged extended supergravity,” Annals Phys. **144**, 249 (1982); “Positive energy in anti-de Sitter backgrounds and gauged extended supergravity,” Phys. Lett. B **115**, 197 (1982).
- [13] L. Mezincescu and P. K. Townsend, “Stability at a local maximum in higher dimensional anti-de Sitter space and applications to supergravity,” Annals Phys. **160**, 406 (1985).
- [14] C. Fefferman and C.R. Graham, “Conformal invariants,” *Élie Cartan et les Mathématiques d’Aujourd’hui* (Astérisque, 1985) 95.
- [15] A. Strominger, “A simple proof of the chiral gravity conjecture,” arXiv:0808.0506 [hep-th].

- [16] K. A. Moussa, G. Clement and C. Leygnac, “The black holes of topologically massive gravity,” *Class. Quant. Grav.* **20**, L277 (2003), arXiv:gr-qc/0303042.
- [17] A. Achúcarro and P.K. Townsend, “A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories,” *Phys. Lett. B* **180**, 89 (1986).
- [18] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system,” *Nucl. Phys. B* **311**, 46 (1988).
- [19] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett. B* **428**, 105 (1998), arXiv:hep-th/9802109.
- [20] D. Grumiller and N. Johansson, “Instability in cosmological topologically massive gravity at the chiral point,” *JHEP* **0807** 134 (2008), arXiv:0805.2610 [hep-th].
- [21] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, 1973), section 6.8.
- [22] W. Li, W. Song, and A. Strominger, “Comment on ‘Cosmological Topological Massive Gravitons and Photons’,” arXiv:0805.3101 [hep-th].
- [23] C. Crnkovic and E. Witten, “Covariant description of canonical formalism in geometric theories,” in *Three hundred years of gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
- [24] J. Lee and R. M. Wald, “Local symmetries and constraints,” *J. Math. Phys.* **31** (1990) 725; R. M. Wald, “Black hole entropy is the Noether charge,” *Phys. Rev.* **D48** (1993) R3427.
- [25] P. Baekler, E. W. Mielke and F. W. Hehl, “Dynamical symmetries in topological 3-D gravity with torsion,” *Nuovo Cim. B* **107**, 91 (1992).
- [26] S. Cacciatori, M. Caldarelli, A. Giacomini, D. Klemm, and D. S. Mansi, “Chern-Simons formulation of three-dimensional gravity with torsion and nonmetricity,” *J. Geom. Phys.* **56**, 2523 (2006), arXiv:hep-th/0507200.

- [27] J. H. Horne and E. Witten, “Conformal gravity in three dimensions as a gauge theory,” *Phys. Rev. Lett.* **62**, 501 (1989).
- [28] S. Carlip, “Quantum gravity in 2+1 dimensions: the case of a closed universe,” *Living Rev. Rel.* **8**, 1 (2005), arXiv:gr-qc/0409039.
- [29] S. Deser and X. Xiang, “Canonical formulations of full nonlinear topologically massive gravity,” *Phys. Lett. B* **263**, 39 (1991).
- [30] S. Carlip, “Inducing Liouville theory from topologically massive gravity,” *Nucl. Phys.* **B362**, 111 (1991).
- [31] S. Deser and R. Jackiw, “Higher derivative Chern-Simons extensions,” *Phys. Lett. B* **451**, 73 (1999), arXiv:hep-th/9901125.
- [32] S. Deser, “Distended Topologically Massive Electrodynamics,” to appear in Wolfgang Kummer memorial volume, arXiv:0810.5384 [hep-th].
- [33] S. Carlip, “The constraint algebra of topologically massive AdS gravity,” *JHEP* **0810** 078 (2008) arXiv:0807.4152 [hep-th].
- [34] D. Grumiller, R. Jackiw, and N. Johansson, “Canonical analysis of cosmological topologically massive gravity at the chiral point,” arXiv:0806.4185 [hep-th].
- [35] G. Giribet, M. Kleban, and M. Porrati, “Topologically massive gravity at the chiral point is not unitary,” *JHEP* **0810** 045 (2008), arXiv:0807.4703v2 [hep-th].
- [36] V. K. Dobrev, “Intertwining operator realization of the AdS/CFT correspondence,” *Nucl. Phys. B* **533**, 559 (1999), arXiv:hep-th/9812194; S. Deser and A. Waldron, “Arbitrary spin representations in de Sitter from dS/CFT with applications to dS supergravity,” *Nucl. Phys. D* **662**, 379 (2003), arXiv:hep-th/0301068.
- [37] G. W. Gibbons, C. N. Pope, and E. Sezgin, “The General Supersymmetric Solution of Topologically Massive Supergravity,” *Class. Quant. Grav.* **25**, 205005 (2008), arXiv:0807.2613 [hep-th].  
A. Garbarz, G. Giribet, and Y. Vásquez, “Asymptotically AdS<sub>3</sub> Solutions to Topologically Massive Gravity at Special Values of the Coupling Constants,” arXiv:0811.4464 [hep-th].

M. Henneaux, C. Martinez, and R. Troncoso, “Asymptotically anti-de Sitter spacetimes in topologically massive gravity,” arXiv:0901.2874 [hep-th].

E. A. Bergshoeff, O. Hohm, and P. K. Townsend, “Massive Gravity in Three Dimensions,” arXiv:0901.1766 [hep-th].